Advanced Logic 2012–13

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week 2

this week

- alternative view on semantics
- substitution preserves validity
- modal definability of frame properties
- bisimulations
- modal truth is invariant under bisimulations

local truth

Let $\mathfrak{M} = (W, R, V)$ be a model, w a point of the model, and φ a modal formula. We define $\mathfrak{M}, w \vDash \varphi$ inductively by:

$$
\begin{array}{lcl}
\mathfrak{M}, w \vDash p & \iff & w \in V(p) \\
\mathfrak{M}, w \vDash \bot & & \text{never} \\
\mathfrak{M}, w \vDash \top & & \text{always} \\
\mathfrak{M}, w \vDash \neg\psi & \iff & \mathfrak{M}, w \nvDash \psi \,, \\
\mathfrak{M}, w \vDash \psi \vee \xi & \iff & \mathfrak{M}, w \vDash \psi \text{ or } \mathfrak{M}, w \vDash \xi \\
\mathfrak{M}, w \vDash \psi \wedge \xi & \iff & \mathfrak{M}, w \vDash \psi \text{ and } \mathfrak{M}, w \vDash \xi \\
\mathfrak{M}, w \vDash \psi \rightarrow \xi & \iff & \mathfrak{M}, w \nvDash \psi \text{ or } \mathfrak{M}, w \vDash \xi \\
\mathfrak{M}, w \vDash \Diamond\psi & \iff & \mathfrak{M}, v \vDash \psi \text{ for some } v \text{ with } Rwv \\
\mathfrak{M}, w \vDash \Box\psi & \iff & \mathfrak{M}, v \vDash \psi \text{ for all } v \text{ with } Rwv
\end{array}
$$

three levels of semantics

- **local truth**:

  $\mathfrak{M}, w \models \varphi$ — φ is **true in point** w of model $\mathfrak{M}$

- **global truth**:

  $\mathfrak{M} \models \varphi$ — φ is **true throughout model** $\mathfrak{M}$:
  $\mathfrak{M}, w \models \varphi$ for all points w of $\mathfrak{M}$

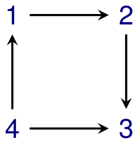
- **validity**:

  $\mathfrak{F} \models \varphi$ — φ is **valid in frame** $\mathfrak{F}$:
  $(\mathfrak{F}, V) \models \varphi$ for all valuations V

  $\mathsf{K} \models \varphi$ — φ is **valid in frame class** K:
  $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F} \in \mathsf{K}$

  $\models \varphi$ — φ is **universally valid**, φ is a **tautology**:
  $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F}$

modal formulas distinguishing states



- find for every point i a **distinguishing** formula φ_i using only the constants $\bot$ and $\top$, that is, formulas φ_i such that:

$$i \models \varphi_j \quad \text{if and only if} \quad i = j \tag{$\sharp$}$$

  for all $i, j \in \{1, 2, 3, 4\}$
- we found the following:

$$\varphi_3 = \Box\bot \qquad \varphi_4 = \Diamond\Diamond\Diamond\top$$
$$\varphi_2 = \Box\Box\bot \wedge \Diamond\top \qquad \varphi_1 = \Diamond\Diamond\Box\bot$$

  and, for these formulas, we proved both directions of ($\sharp$)
- of course there are (infinitely) many alternatives. for instance we could have taken $\varphi_1 = \Diamond\varphi_2$, for 1 is the only point that sees state 2 which was already uniquely determined by φ_2

game semantics

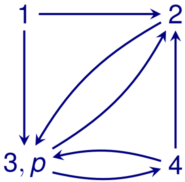
- let $\mathfrak{M}, s$ be a pointed Ω-model, and φ a formula over Ω
- two players:
  - ***V*** (initially) claims that φ is true at s
  - ***F*** (initially) claims that φ is false at s
- start at **position** (s, φ)
- at position (t, ψ) the next move is determined by the main connective of ψ, as follows[1]:

$(t, \psi_1 \vee \psi_2)$	A chooses a disjunct ψ_i, play continues with (t, ψ_i)
$(t, \psi_1 \wedge \psi_2)$	B chooses a conjunct ψ_i, play continues with (t, ψ_i)
$(t, \Diamond\psi_0)$	A chooses a successor u of t, play continues with (u, ψ_0)
$(t, \Box\psi_0)$	B chooses a successor u of t, play continues with (u, ψ_0)
$(t, \neg\psi_0)$	players switch roles (colors), play continues with (t, ψ_0)
(t, p)	if $\mathfrak{M}, s \models p$ then A wins
	if $\mathfrak{M}, s \not\models p$ then B wins
$(t, \top)$	A wins
$(t, \bot)$	B wins

[1] as primitive connectives we here take $\bot, \top, \neg, \vee, \wedge, \Diamond, \Box$

game semantics

- a **strategy** for player P is a method P uses to select her moves
- P has a **winning** strategy for a game if the strategy ensures that every possible match of the game (path through the tree) is won by P
- consider the model



and the modal formula $\varphi = \Box(\Diamond p \vee \Box\Diamond p)$
- draw the complete game tree starting at position $(1, \varphi)$
- who has a winning strategy for this game?
- theorem: ***V*** has a winning strategy for a game $(\mathfrak{M}, s, \varphi)$ if and only if $\mathfrak{M}, s \models \varphi$

an alternative semantics

Definition

Let $\mathfrak{M} = (W, R, V)$ be a model. We define $[\![\varphi]\!]_{\mathfrak{M}} \subseteq W$, the **interpretation** of a formula φ in the model $\mathfrak{M}$, inductively by

$$
\begin{aligned}
[\![p]\!]_{\mathfrak{M}} &= V(p) && (p \in \Omega)\\
[\![\bot]\!]_{\mathfrak{M}} &= \varnothing\\
[\![\top]\!]_{\mathfrak{M}} &= W\\
[\![\neg\varphi]\!]_{\mathfrak{M}} &= W \setminus [\![\varphi]\!]_{\mathfrak{M}}\\
[\![\varphi \vee \psi]\!]_{\mathfrak{M}} &= [\![\varphi]\!]_{\mathfrak{M}} \cup [\![\psi]\!]_{\mathfrak{M}}\\
[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} &= [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}\\
[\![\varphi \to \psi]\!]_{\mathfrak{M}} &= (W \setminus [\![\varphi]\!]_{\mathfrak{M}}) \cup [\![\psi]\!]_{\mathfrak{M}}\\
[\![\Diamond\varphi]\!]_{\mathfrak{M}} &= \{\, w \in W \mid \exists v\,(Rwv \wedge v \in [\![\varphi]\!]_{\mathfrak{M}})\,\}\\
[\![\Box\varphi]\!]_{\mathfrak{M}} &= \{\, w \in W \mid \forall v\,(Rwv \implies v \in [\![\varphi]\!]_{\mathfrak{M}})\,\}
\end{aligned}
$$

an alternative semantics

Lemma

The interpretation of φ in $\mathfrak{M} = (W, R, V)$ is the set of states of $\mathfrak{M}$ where φ is true:

$$\mathfrak{M}, w \models \varphi \iff w \in [\![\varphi]\!]_{\mathfrak{M}}$$

Hence, we also have:

$$\mathfrak{M} \models \varphi \iff [\![\varphi]\!]_{\mathfrak{M}} = W \tag{1}$$

evaluating substitution instances

Definition

Let $\mathfrak{M} = (W, R, V)$ be a model, φ a formula, and σ a substitution.
Define $\mathfrak{M}^\sigma = (W, R, V^\sigma)$ where V^σ is defined by

$$(V^\sigma)(p) = [\![\sigma(p)]\!]_{\mathfrak{M}} \qquad (p \in \Omega)$$

Lemma

$$[\![\varphi^\sigma]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}^\sigma} \tag{2}$$

validity is closed under substitution

Theorem

If a formula φ is valid in a frame $\mathfrak{F}$, then so are all its substitution instances:

$$\mathfrak{F} \models \varphi \implies \mathfrak{F} \models \varphi^{\sigma}$$

Proof: immediate from (1) and (2).

substitution as a tool to derive new validities

- substitution generates new valid formulas from old ones
- example: if $\mathfrak{F} \models \Diamond\Box p \to \Box\Diamond p$ then $\mathfrak{F} \models \Diamond\Box\varphi \to \Box\Diamond\varphi$ for all formulas φ
- if ψ is a substitution instance of a propositional tautology then ψ is a modal tautology
- let $\delta : \Omega \to \mathit{FORM}(\Omega)$ be the substitution defined by

$$\delta(p) = \neg p \qquad\qquad (p \in \Omega)$$

- then we have:

$$\mathfrak{F} \models \varphi \iff \mathfrak{F} \models \varphi^\delta$$

- and so:

$$\begin{aligned}
\mathfrak{F} \models p \to \Diamond p &\iff \mathfrak{F} \models \Box p \to p \\
\mathfrak{F} \models \Diamond\Diamond p \to \Diamond p &\iff \mathfrak{F} \models \Box p \to \Box\Box p \\
\mathfrak{F} \models p \to \Box\Diamond p &\iff \mathfrak{F} \models \Diamond\Box p \to p \\
\mathfrak{F} \models \Diamond p \to \Box\Diamond p &\iff \mathfrak{F} \models \Diamond\Box p \to \Box p
\end{aligned}$$

etc.

$\Diamond\Box p \to p$ valid in all symmetric frames

let $\text{Symm} = \{(W, R) \mid \forall xy\,(Rxy \to Ryx)\}$

$$\text{Symm} \models \Diamond\Box p \to p$$

1. let $\mathfrak{F} = (W, R) \in \text{Symm}$
2. consider an arbitrary model $\mathfrak{M}$ based on $\mathfrak{F}$, and a state $w \in W$ such that $\mathfrak{M}, w \models \Diamond\Box p$. we have to show $\mathfrak{M}, w \models p$
3. from 2 follows the existence of a state $v \in W$ such that
   a. Rwv and
   b. $\mathfrak{M}, v \models \Box p$
4. $\mathfrak{M}, u \models p$ for all u with Rvu (3b)
5. Rvw (3a & 1)
6. $\mathfrak{M}, w \models p$ (5 & 4)
7. $\mathfrak{M}, w \models \Diamond\Box p \to p$ (2 & 6)
8. $\text{Symm} \models \Diamond\Box p \to p$

truth versus validity

- validity is a **stable** form of truth
- validity is independent of valuation, truth is not, e.g. :
  - $\mathfrak{F} \models p \to \Diamond p$ implies $\mathfrak{F} \models q \to \Diamond q$
  - but $\mathfrak{M} \models p \to \Diamond p$ does **not** imply $\mathfrak{M} \models q \to \Diamond q$

- local truth (truth in a point of a model) is preserved by
  - modus ponens: if $\mathfrak{M}, w \models \varphi \to \psi$ en $\mathfrak{M}, w \models \varphi$ the $\mathfrak{M}, w \models \psi$
- global truth (truth in all points of a model) is preserved by
  - modus ponens: if $\mathfrak{M} \models \varphi \to \psi$ en $\mathfrak{M} \models \varphi$ the $\mathfrak{M} \models \psi$
  - necessitation: if $\mathfrak{M} \models \varphi$ then $\mathfrak{M} \models \Box\varphi$
- frame validity (truth in all models on the frame) is preserved by:
  - modus ponens: if $\mathfrak{F} \models \varphi \to \psi$ and $\mathfrak{F} \models \varphi$ then $\mathfrak{F} \models \psi$
  - necessitation: if $\mathfrak{F} \models \varphi$ then $\mathfrak{F} \models \Box\varphi$
  - substitution: if $\mathfrak{F} \models \varphi$ then $\mathfrak{F} \models \varphi^\sigma$

universal validity

Proposition

The set of universally valid formulas, or modal tautologies:

- *contains all propositional tautologies*
- *contains* $\Box(p \to q) \to (\Box p \to \Box q)$
- *is closed under modus ponens:*
  if $\vDash \varphi \to \psi$ *and* $\vDash \varphi$ *then* $\vDash \psi$
- *is closed under necessitation:*
  if $\vDash \varphi$ *then* $\vDash \Box\varphi$
- *is closed under substitution:*
  if $\vDash \varphi$ *then* $\vDash \varphi^\sigma$
- *and contains nothing else.*

modal definability of frame properties

Definition

A modal formula φ **defines**, or **characterizes**, a class K of frames (a frame property) when

$$\mathfrak{F} \in \mathsf{K} \iff \mathfrak{F} \models \varphi$$

In other words, φ characterizes K if

($\Rightarrow$) φ is valid in K:

$$\mathsf{K} \models \varphi$$

and

($\Leftarrow$) φ is invalid outside of K:

$$\text{if } \mathfrak{F} \notin \mathsf{K} \text{ then } \mathfrak{F} \not\models \varphi$$

example: $\Diamond\Box p \to p$ characterizes symmetry

Example

$$\mathfrak{F} \in \text{Symm} \quad \text{if and only if} \quad \mathfrak{F} \models \Diamond\Box p \to p$$

($\Rightarrow$) Symm $\models \Diamond\Box p \to p$ (slide 13)

($\Leftarrow$) by contraposition:

- let $\mathfrak{F} = (W, R) \notin$ Symm. we will show $\mathfrak{F} \not\models \Diamond\Box p \to p$
- as R is not symmetric, there are states a and b (not necessarily distinct) such that Rab and $\neg Rba$
- let V on $\mathfrak{F}$ be such that $V(p) = \{ x \in W \mid Rbx \}$ and $\mathfrak{M} = (\mathfrak{F}, V)$
- then $\mathfrak{M}, b \models \Box p$
- by Rab also $\mathfrak{M}, a \models \Diamond\Box p$
- also $\mathfrak{M}, a \not\models p$, as $a \notin V(p)$ since $\neg Rba$
- so $\mathfrak{M}, a \not\models \Diamond\Box p \to p$ and so $\mathfrak{F} \not\models \Diamond\Box p \to p$
- conclusion: if $\Diamond\Box p \to p$ is valid in a frame, then the frame is symmetric

example: $\Diamond p \to \Box p$ characterizes partial functionality

Example

$$\mathfrak{F} \in \mathrm{PF} \quad \text{if and only if} \quad \mathfrak{F} \models \Diamond p \to \Box p$$

where

$$\mathrm{PF} = \{\, (W, R) \mid \forall xyz\, (Rxy \wedge Rxz \to y = z)\,\}$$

example: a modal formula for right-linearity

Example

$$\mathfrak{F} \models \Box((p \wedge \Box p) \to q) \vee \Box((q \wedge \Box q) \to p)$$

if and only if

$\mathfrak{F} = (W, R)$ is right-linear:

$$(Rxy \ \wedge \ Rxz) \implies (Ryz \ \vee \ y = z \ \vee \ Rzy)$$

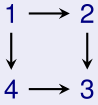
(Similar to item 3 in the first assignment).

towards bisimulations

- what can be expressed by the modal language?
- when can two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ be distinguished by the modal language?
- when should they be viewed as modally identical?
- what is the right semantic equivalence for the basic modal language?

indistinguishable states

Example



- states 2 and 4 cannot be distinguished by a modal formula
- in other words $2 \vDash \varphi$ if and only if $4 \vDash \varphi$, for all formulas φ
- why?

bisimulations

Definition

Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be models.
A non-empty relation $Z \subseteq W \times W'$ is a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$, notation $Z : \mathfrak{M} \leftrightarroweq \mathfrak{M}'$, if for all pairs $(w, w') \in Z$:

- (**base**) $w \in V(p)$ if and only if $w' \in V'(p)$
- (**zig**) if Rwv then for some $v' \in W'$ we have: $R'w'v'$ and $(v, v') \in Z$
- (**zag**) if $R'w'v'$ then for some $v \in W$ we have: Rwv and $(v, v') \in Z$

So bisimilar states carry the **same atomic information**, and whenever it is possible to make a transition in one model, it is possible to make a **matching transition** in the other.

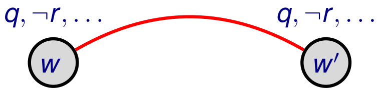
bisimulations: base condition

$\mathfrak{M} = (W, R, V)$

$\mathfrak{M}' = (W', R', V')$

if wZw'

bisimulations: base condition

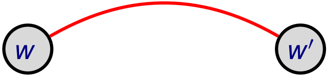


$\mathfrak{M} = (W, R, V)$

$\mathfrak{M}' = (W', R', V')$

if $w Z w'$
then for all $p \in \Omega$
$w \in V(p)$ if and only if $w' \in V'(p)$

bisimulations: the zig-condition

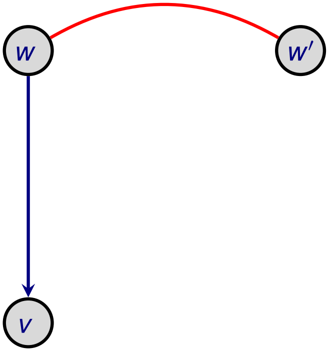


$\mathfrak{M} = (W, R, V)$

$\mathfrak{M}' = (W', R', V')$

if $w\color{red}{Z}w'$

bisimulations: the zig-condition

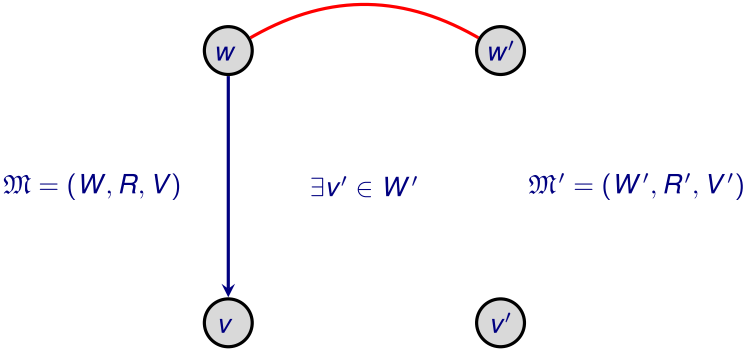


$\mathfrak{M} = (W, R, V)$

$\mathfrak{M}' = (W', R', V')$

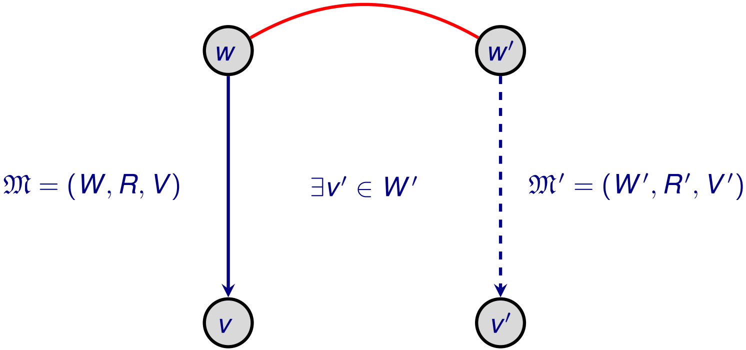
if wZw' and Rwv

bisimulations: the zig-condition



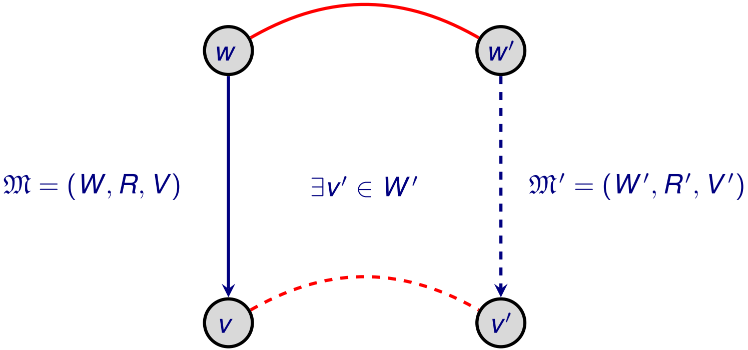
if wZw' and Rwv
then there exists a point $v' \in W'$

bisimulations: the zig-condition



if wZw' and Rwv
then there exists a point $v' \in W'$
such that $R'w'v'$

bisimulations: the zig-condition



if wZw' and Rwv
then there exists a point $v' \in W'$
such that $R'w'v'$ and vZv'

bisimulations: the zag-condition

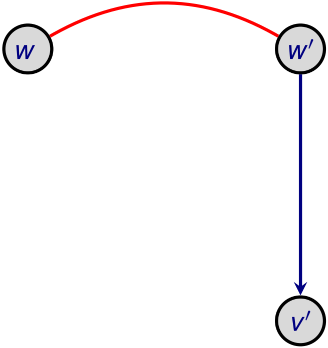
$\mathfrak{M} = (W, R, V)$

$\mathfrak{M}' = (W', R', V')$

if $w Z w'$

bisimulations: the zag-condition

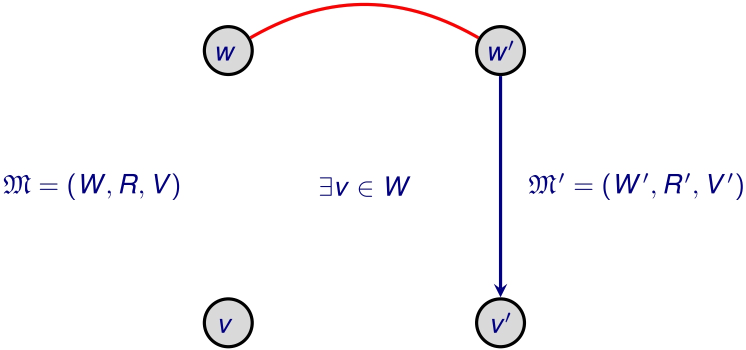
$\mathfrak{M} = (W, R, V)$



$\mathfrak{M}' = (W', R', V')$

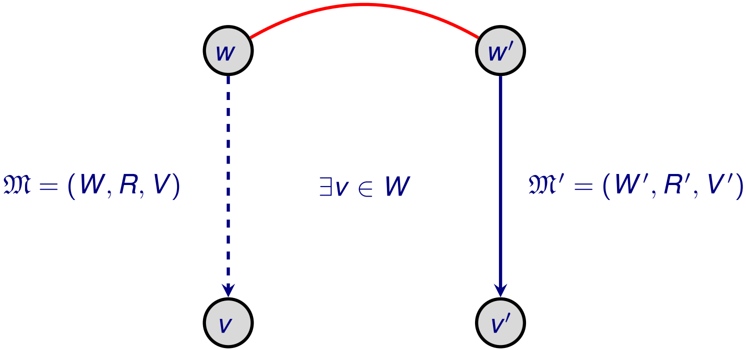
if $w \textcolor{red}{Z} w'$ and $R' w' v'$

bisimulations: the zag-condition



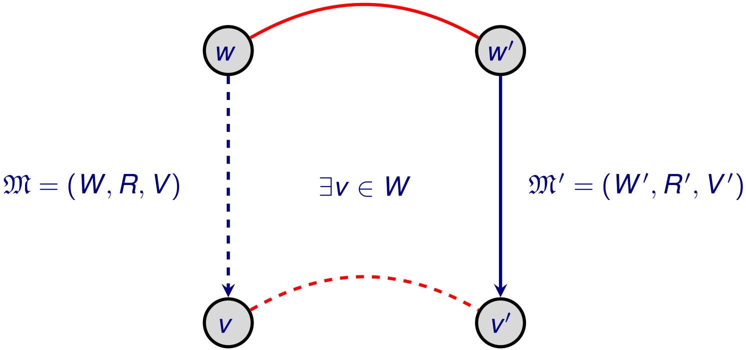
if wZw' and $R'w'v'$
then there exists a point $v \in W$

bisimulations: the zag-condition



if wZw' and $R'w'v'$
then there exists a point $v \in W$
such that Rwv

bisimulations: the zag-condition



if wZw' and $R'w'v'$
then there exists a point $v \in W$
such that Rwv and vZv'

bisimilarity

Definition

Two models $\mathfrak{M}$ and $\mathfrak{M}'$ are **bisimilar**, notation $\mathfrak{M} \underline{\leftrightarrow} \mathfrak{M}'$, if there exists a bisimulation Z such that $Z : \mathfrak{M} \underline{\leftrightarrow} \mathfrak{M}'$.
Two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ are **bisimilar**, notation: $\mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{M}', w'$ or just $w \underline{\leftrightarrow} w'$, if $Z : \mathfrak{M} \underline{\leftrightarrow} \mathfrak{M}'$ and wZw' for some bisimulation Z.

bisimilarity

Proposition

$\underline{\leftrightarrow}$ as a relation between models, is an equivalence relation:

- $\Delta : \mathfrak{M} \underline{\leftrightarrow} \mathfrak{M}$ *where* $\Delta = \{ (w, w) \mid w \in W \}$.
- *If* $Z : \mathfrak{M} \underline{\leftrightarrow} \mathfrak{M}'$, *then* $Z^{-1} : \mathfrak{M}' \underline{\leftrightarrow} \mathfrak{M}$, *where* $Z^{-1} = \{ (w', w) \mid (w, w') \in Z \}$.
- *If* $Z_1 : \mathfrak{M}_1 \underline{\leftrightarrow} \mathfrak{M}_2$ *and* $Z_2 : \mathfrak{M}_2 \underline{\leftrightarrow} \mathfrak{M}_3$ *then* $Z_1 \circ Z_2 : \mathfrak{M}_1 \underline{\leftrightarrow} \mathfrak{M}_3$ *where* $Z_1 \circ Z_2 = \{ (x, z) \mid \exists y \, (xZ_1 y \wedge yZ_2 z) \}$.

example of bisimilar states

Example

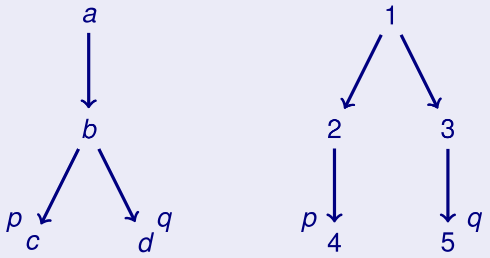
$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 4 & \longrightarrow & 3 \end{array}$$

States 2 and 4 are bisimilar, since there are bisimulations relating them, for example:

$$\begin{aligned} B_1 &= \{(2,4),(3,3)\} \\ B_2 &= \{(1,1),(2,4),(4,2),(3,3)\} \\ B_3 &= \{(1,1),(2,2),(2,4),(3,3),(4,2),(4,4)\} \end{aligned}$$

example of non-bisimilarity

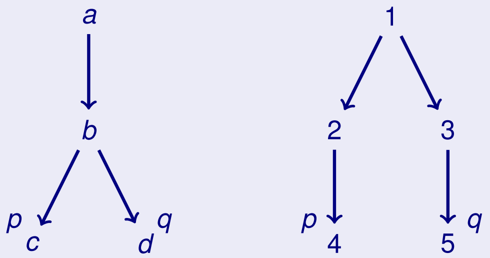
Example



states a and 1 are not bisimilar ...

example of non-bisimilarity

Example

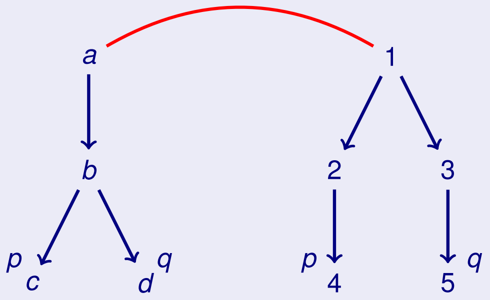


for suppose they were. then $(a, 1) \in Z$ for some bisimulation Z

$$Z \subseteq \{a, b, c, d\} \times \{1, 2, 3, 4, 5\}$$

example of non-bisimilarity

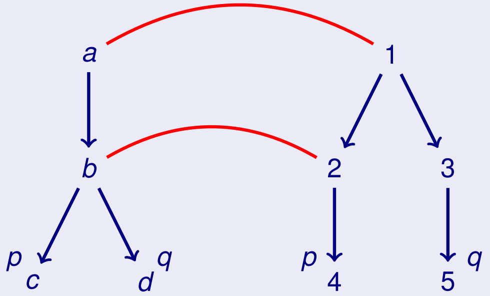
Example



$$Z = \{ (a, 1), \ldots$$

example of non-bisimilarity

Example

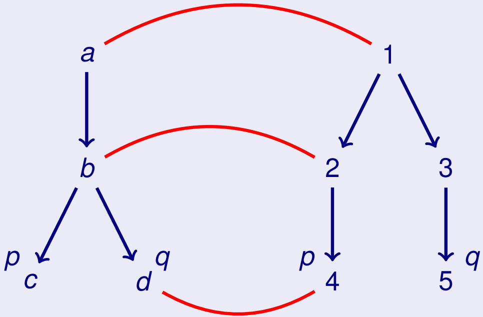


$$Z = \{ (a,1), (b,2) \dots$$

since the step from 1 to 2 has to be matched on the left (zag)

example of non-bisimilarity

Example

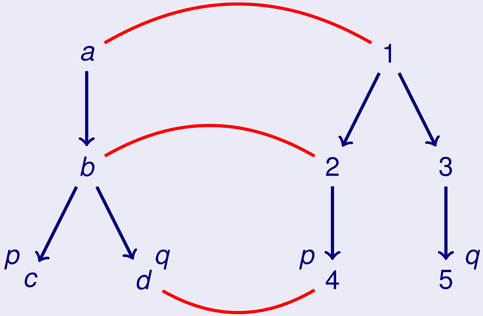


$$Z = \{ (a, 1), (b, 2), (d, 4) \ldots$$

since the step from b to d has to be matched on the right (zig)

example of non-bisimilarity

Example



but d and 4 disagree on their atomic info: $d \not\models p$ whereas $4 \models p$.
hence, there cannot be a bisimulation linking a to 1.

another example of a bisimulation

Example

$$\mathfrak{N} = (\mathbb{N}, S) \qquad \mathfrak{F} = (\{e, o\}, R)$$

$$S = \{(n, n+1) \mid n \in \mathbb{N}\} \qquad R = \{(e, o), (o, e)\}$$

$$V(p) = \{2n \mid n \in \mathbb{N}\} \qquad U(p) = \{e\}$$

State 0 of model $(\mathfrak{N}, V)$ bisimulates with state e of model $(\mathfrak{F}, U)$.

modal equivalence of states

Definition

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be Ω-models.
A state w of $\mathfrak{M}$ and a state w' of $\mathfrak{M}'$ are **modally equivalent**, notation $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$, if they satisfy the same formulas:

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w' \quad \text{if and only if} \quad \forall \varphi\, (\mathfrak{M}, w \vDash \varphi \text{ iff } \mathfrak{M}', w' \vDash \varphi)$$

invariance: $\underline{\leftrightarrow} \subseteq \leftrightsquigarrow$

Theorem

Bisimilar states are modally equivalent:

$$(\mathfrak{M}, w \,\underline{\leftrightarrow}\, \mathfrak{M}', w') \quad \Longrightarrow \quad (\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w')$$

In other words: modal truth is **invariant** under bisimulation.

bounded morphisms: functional frame-bisimulations

Definition

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be frames.
A function $h : W \to W'$ is a **bounded morphism** if it satisfies

- for all $w, v \in W$, if Rwv then $R'h(w)h(v)$
- for all $w \in W$, $v' \in W'$, if $R'h(w)v'$ then there exists $v \in W$ such that $h(v) = v'$ and Rwv

We write $h : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ if h is a **surjective** bounded morphism from $\mathfrak{F}$ to $\mathfrak{F}'$ (so when the image of h is the entire domain of $\mathfrak{F}'$).
We write $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ if $h : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ for some h, and call $\mathfrak{F}'$ a **bounded morphic image** of $\mathfrak{F}$.

Note that the relation $H = \{(x, h(x)) \mid x \in W\}$ satisfies the zig and zag conditions of bisimulation.

surjective bounded morphisms preserve frame validity

Theorem

A bounded morphic image $\mathfrak{F}'$ of $\mathfrak{F}$ contains the theory of $\mathfrak{F}$, i.e.,

$$(\mathfrak{F} \twoheadrightarrow \mathfrak{F}') \quad \Longrightarrow \quad (\mathfrak{F} \models \varphi \quad \Longrightarrow \quad \mathfrak{F}' \models \varphi)$$

Proof.

Let $h : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ be a bounded morphism, φ a modal formula, and assume $\mathfrak{F} \models \varphi$. Let V' be an arb. val. on $\mathfrak{F}'$, and w' an arb. point of $\mathfrak{F}'$. Define V on $\mathfrak{F}$ by $V(p) = \{x \in \mathrm{dom}(\mathfrak{F}') \mid h(x) \in V'(p)\}$ for all $p \in \Omega$. By surj. of h, $w' = h(w)$ for some point w of $\mathfrak{F}$. From $\mathfrak{F} \models \varphi$ we know $\mathfrak{F}, V, w \models \varphi$. Clearly, $\mathfrak{F}, V, w \,\underline{\leftrightarrow}\, \mathfrak{F}', V', w'$, and so, by invariance under bisimulation, $\mathfrak{F}', V', w' \models \varphi$. $\dashv$

application: asymmetry not modally definable

Example

There is no modal formula that characterizes asymmetry ($Rxy \to \neg Ryx$); proof using frames $\mathfrak{N}$ en $\mathfrak{F}$ from slide 44:

- suppose there was such a formula φ
- then $\mathfrak{N} \models \varphi$
- let h be defined by $h(2n) = e$ and $h(2n+1) = o$
- then $h : \mathfrak{N} \twoheadrightarrow \mathfrak{F}$ and so $\mathfrak{F} \models \varphi$
- contradiction, as $\mathfrak{F}$ is not asymmetric
- hence φ does not exist

In general:

Corollary

Let K *be a class of frames, and let* $\mathfrak{F}, \mathfrak{F}'$ *be frames. If* $\mathfrak{F} \in \mathsf{K}$, $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ *and* $\mathfrak{F}' \notin \mathsf{K}$, *then* K *cannot be characterized bu a modal formula.*

$\underline{\leftrightsquigarrow} \subseteq \underline{\leftrightarrow}$?

What about the other direction: does modal equivalence of states imply that they are bisimilar?

for finitely branching models: $\underline{\leftrightarrow} = \leftrightsquigarrow$!

For finitely branching models (where every state has finitely many successors), bisimilarity and modal equivalence of states coincide!

Theorem

Let $\mathfrak{M}$, and $\mathfrak{M}'$ be finitely branching models.
Then for all states s of $\mathfrak{M}$ and s' of $\mathfrak{M}'$ we have

$$(\mathfrak{M}, s \;\underline{\leftrightarrow}\; \mathfrak{M}', s') \iff (\mathfrak{M}, s \leftrightsquigarrow \mathfrak{M}', s')$$