modal logic: two viewpoints

formal
: view modal logic as a formal system for describing and reasoning with modalities

semantic
: view the modal language as a tool for the analysis of relational structures

modalities

a modality qualifies the truth of an assertion, indicates how an assertion relates to 'reality'

let φ be some assertion. examples of 'diamonds' and 'boxes' are:

$\Diamond\varphi$ = it is possible that φ	$\Box\varphi$ = it is necessary that φ
$\langle F\rangle\varphi$ = at some future time φ will hold	$[F]\varphi$ = henceforth φ holds
$\langle P\rangle\varphi$ = at some past time, φ was true	$[P]\varphi$ = hitherto φ
$P\varphi$ = it is permitted that φ	$O\varphi$ = it is obligatory that φ
$\langle\alpha\rangle\varphi$ = after some run of α, φ holds	$[\alpha]\varphi$ = after each run of α: φ
$\Diamond\varphi$ = φ is consistent with PA	$\Box\varphi$ = φ is provable in PA
$\langle A\rangle\varphi$ = A considers φ possible	$K_A\varphi$ = A knows (for sure) that φ

note the duality between the respective diamonds and boxes:

$$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi \quad \text{and} \quad \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$$

which rules and axioms?

($\Box$) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(N) if φ is derivable, then so is $\Box\varphi$

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(T)	$\Box\varphi \to \varphi$	$O\varphi \to \varphi$
(4)	$\Box\varphi \to \Box\Box\varphi$	$K_i\,\varphi \to K_i\,K_i\,\varphi$
(5)	$\Diamond\varphi \to \Box\Diamond\varphi$	$\neg K_i\,\varphi \to K_i\,\neg K_i\,\varphi$
(B)	$\varphi \to \Box\Diamond\varphi$	$\varphi \to [F]\langle P\rangle\varphi$
		$\varphi \to [P]\langle F\rangle\varphi$
(D)	$\Box\varphi \to \Diamond\varphi$	$O\varphi \to P\varphi$

the need for semantics

- what assumptions do we make about time, knowledge, etc.?
- how do we know that a certain candidate axiom is new (and not already derivable)?
- how do we know that our system is consistent?
- a proof system for a language is a collection of axioms and rules designed intended to prove precisely the valid statements expressible in the language:
  - the system has to be sound: all derived statements must be valid, and
  - complete: every valid statement can be proved

so we need a precise notion of validity!

this will employ: possible worlds semantics, or relational semantics

semantic perspective

modal languages are simple but expressive languages to talk about relational structures (graphs), viewed locally and from the inside

- truth is relative to the current situation (moment, state, etc.)
- modal formulas are evaluated within a given structure and from within a specific state
- modal operators scan (analyze, collect) information only of states which are accessible via a transition step from the current state
- necessity = truth in all accessible states, in all possible worlds
- possibility = truth in at least one accessible state

semantic perspective

study the properties of the transition relation R:

- do we choose
  - $R = W \times W$?
  - $R = \varnothing$?
  - R reflexive, transitive, . . . ?
- what is the impact of these choices on notions like possibility and necessity?
- which principles are justified if we demand that R is symmetric? and which are excluded?
- which structural properties can be distinguished by the modal language? when does it consider two structures equivalent?

propositional logic: language

- let $\Omega = \{p, q, \ldots, p_0, p_1, p_2, \ldots\}$ be a set of propositional variables
- the set $PROP(\Omega)$ of propositional formulas over Ω is defined by

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi_0 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \to \varphi_2$$

  where $p \in \Omega$
- in words: $PROP(\Omega)$ is the smallest set X (wrt $\subseteq$) such that:
  1. $\Omega \subseteq X$ and $\bot, \top \in X$
  2. X is closed under $\neg$: if $\varphi \in X$, then $\neg\varphi \in X$
  3. X is closed under $* \in \{\vee, \wedge, \to\}$: if $\varphi, \psi \in X$, then $\varphi * \psi \in X$
- note: 'smallest' ensures that 1,2,3 are the only ways to build elements of $PROP(\Omega)$
- we let $\neg$ bind stronger than $\vee, \wedge$ stronger than $\to, \leftrightarrow$

propositional logic: language

- sometimes we take a (functionally complete) subset of the above connectives to be primitive, and define the others in terms of these primitives; for example we let

  $$\varphi ::= p \mid \bot \mid \varphi_1 \to \varphi_2$$

  and define

  $$\begin{aligned} \neg\varphi &:= \varphi \to \bot \\ \top &:= \neg\bot \\ \varphi \vee \psi &:= \neg\varphi \to \psi \\ \varphi \wedge \psi &:= \neg(\varphi \to \neg\psi) \\ &\text{etc.} \end{aligned}$$

  (equivalent in the standard semantics, next slide)
- reason: less cases when defining or proving things by induction

propositional logic: semantics

- associate to each connective an operation on $\mathbf{2} = \{0, 1\}$
- we use an operation $\to : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ defined by

x	y	$x \to y$
0	0	1
0	1	1
1	0	0
1	1	1

  to interpret the arrow connective $\to$
- a valuation is a map $\pi : \Omega \to \mathbf{2}$ assigning truth values to variables
- π induces a map $[\![\cdot]\!]_\pi : \mathit{PROP}(\Omega) \to \mathbf{2}$

$$
\begin{aligned}
[\![p]\!]_\pi &= \pi(p) \\
[\![\bot]\!]_\pi &= 0 \\
[\![\varphi \to \psi]\!]_\pi &= [\![\varphi]\!]_\pi \to [\![\psi]\!]_\pi
\end{aligned}
$$

- the semantic clauses for the other connectives are derived
- we say $[\![\varphi]\!]_\pi$ is the interpretation of φ under π

propositional logic: two-valued semantics

- φ is true under π or π is a model for φ if $[\![\varphi]\!]_\pi = 1$
- φ is a semantic consequence of $\Gamma \subseteq \mathit{PROP}(\Omega)$:

$$\Gamma \vDash \varphi$$

  iff every model for (every formula of) Γ is a model for φ:

$$(\forall \psi \in \Gamma\,([\![\psi]\!]_\pi = 1)) \implies ([\![\varphi]\!]_\pi = 1)$$

  for all assignments $\pi : \Omega \to \mathbf{2}$
- φ is a tautology, notation $\vDash \varphi$, iff it is true in all models:

$$\vDash \varphi \iff \varnothing \vDash \varphi$$

  so φ is a tautology iff $[\![\varphi]\!]_\pi = 1$, for all valuations $\pi : \Omega \to \mathbf{2}$
- beware: the following implications do NOT hold:

$$(\Gamma \nvDash \varphi) \implies (\Gamma \vDash \neg\varphi)$$
$$(\Gamma \vDash \varphi \vee \psi) \implies (\Gamma \vDash \varphi) \text{ or } (\Gamma \vDash \psi)$$

propositional logic: proof systems

goal: a syntactic description of semantic consequence

- define a (Hilbert style) proof system H, consisting of:
  - three axiom schemes:

$$\varphi \to (\psi \to \varphi) \qquad K$$
$$(\varphi \to (\psi \to \xi)) \to ((\varphi \to \psi) \to (\varphi \to \xi)) \qquad S$$
$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi) \qquad C$$

  - one derivation rule, modus ponens:
    if $\varphi \to \psi$ and φ are derivable, then ψ is derivable:
- an H-proof of φ from a set of formulas Γ (hypotheses) is a sequence $\varphi_1, \ldots, \varphi_n = \varphi$ such that for all φ_i ($1 \leq i \leq n$) one of the following holds:

(hyp) $\varphi_i \in \Gamma$
(ax) φ_i is an axiom (an instance of one of the above schemes)
(mp) there are φ_j, φ_k with $j, k < i$ such that $\varphi_k = \varphi_j \to \varphi_i$

- we write $\Gamma \vdash \varphi$ if there is an H-proof of φ from Γ

soundness and completeness

this Hilbert-style proof system is sound and complete:

sound: provable statements are true:

$$\Gamma \vdash \varphi \implies \Gamma \models \varphi$$

complete: true statements are provable:

$$\Gamma \models \varphi \implies \Gamma \vdash \varphi$$

basic modal logic: language

- let $\Omega = \{p, q, \ldots, p_0, p_1, p_2, \ldots\}$ be a set of propositional variables
- the set $FORM(\Omega)$ of basic modal formulas (over Ω) is defined by

  $$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi_0 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \Diamond\varphi_0 \mid \Box\varphi_0$$

  where $p \in \Omega$
- we use $\varphi, \psi, \xi, \ldots, \varphi_0, \varphi_1, \ldots$ to range over formulas
- constants $\bot$, $\top$ and variables $p \in \Omega$ are called atomic formulas
- binding: we let the unary operators $\neg, \Diamond, \Box$ bind stronger than $\vee, \wedge$, which in turn bind stronger than $\rightarrow, \leftrightarrow$
  example:

  $\Box(\neg\psi \wedge \varphi) \rightarrow \Diamond\psi \vee \varphi$ is parsed as $\Box((\neg\psi) \wedge \varphi) \rightarrow ((\Diamond\psi) \vee \varphi)$

induction on structure of formulas

- we obtain the principle of induction on the structure of formulas:

  a property $P \subseteq FORM(\Omega)$ holds for all formulas ($P = FORM(\Omega)$) if P contains all atomic formulas, and is closed under the connectives

- this means that, writing $P(\varphi)$ for $\varphi \in P$, in order to prove $P(\varphi)$ for all formulas φ, it suffices to show:
  - $P(\bot)$, $P(\top)$, and $P(p)$, for all $p \in \Omega$
  - if $P(\psi)$, then $P(\neg\psi)$
  - if $P(\psi_1)$ and $P(\psi_2)$, then $P(\psi_1 * \psi_2)$ for each $* \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$
  - if $P(\psi)$, then $P(\Diamond\psi)$
  - if $P(\psi)$, then $P(\Box\psi)$
- (as before) sometimes we take a subset of the connectives to be primitive, and consider the others to be defined. for example we may take $\{\top, \neg, \vee, \Diamond\}$, or $\{\bot, \rightarrow, \Box\}$.

frames and models

- a frame is a pair $\mathfrak{F} = (W, R)$ where W is a non-empty set of states, called the domain of $\mathfrak{F}$, and $R \subseteq W \times W$ is a binary relation on W, called the accessibility or transition relation of $\mathfrak{F}$
- a Ω-model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (W, R)$ is a frame, and V is a valuation (on $\mathfrak{F}$), that is, a function $V : \Omega \to \mathbf{2}^W$ assigning to each atomic formula $p \in \Omega$ a subset $V(p)$ of W. we say that $\mathfrak{M}$ is a model (based) on $\mathfrak{F}$

  ($\mathbf{2}^A$ is the powerset of A, that is, the set of all subsets of A: $X \in \mathbf{2}^A$ iff $X \subseteq A$)

- a pointed Ω-model is a pair $(\mathfrak{M}, w)$ with $\mathfrak{M}$ a Ω-model and w a state in $\mathfrak{M}$

truth definition

- for a Ω-model $\mathfrak{M} = (W, R, V)$, a point $w \in W$, and a formula $\varphi \in FORM(\Omega)$ we define $\mathfrak{M}, w \models \varphi$ inductively by:

$\mathfrak{M}, w \models p$	iff	$w \in V(p)$
$\mathfrak{M}, w \models \bot$		never
$\mathfrak{M}, w \models \varphi \to \psi$	iff	$\mathfrak{M}, w \models \varphi$ implies $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \Box\varphi$	iff	for all v with Rwv we have $\mathfrak{M}, v \models \varphi$

- if $\mathfrak{M}, w \models \varphi$ we say w satisfies φ (in $\mathfrak{M}$), or φ is true at w (in $\mathfrak{M}$)
- if $\mathfrak{M}, w$ does not satisfy φ we write $\mathfrak{M}, w \not\models \varphi$, e.g.

$$\mathfrak{M}, w \models \varphi \to \psi \quad \text{iff} \quad \mathfrak{M}, w \not\models \varphi \text{ or } \mathfrak{M}, w \models \psi$$

derived truth definitions

for the defined connectives we find

$\mathfrak{M}, w \models \top$		always
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \vee \psi$	iff	$\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \leftrightarrow \psi$	iff	$\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \Diamond\varphi$	iff	$\mathfrak{M}, v \models \varphi$ for some v with Rwv

furthermore, note that:

$\mathfrak{M}, w \not\models \Box\varphi$	iff	for some $v \in W$ with Rwv we have $\mathfrak{M}, v \not\models \varphi$
$\mathfrak{M}, w \not\models \Diamond\varphi$	iff	for all $v \in W$ with Rwv we have $\mathfrak{M}, v \not\models \varphi$
$\mathfrak{M}, w \models \Diamond\top$	iff	for some v we have Rwv
$\mathfrak{M}, w \models \Box\bot$	iff	there is no v such that Rwv
$\mathfrak{M}, w \not\models \Diamond\bot$		always
$\mathfrak{M}, w \models \Box\top$		always

further terminology

- if φ holds at some state in $\mathfrak{M}$, we say that φ is satisfiable in $\mathfrak{M}$
- if φ holds at some state in some model, we say that φ is satisfiable
- let $(\mathfrak{M}_1, w_1)$ and $(\mathfrak{M}_2, w_2)$ be pointed Ω-models. we say that a formula φ over Ω distinguishes $(\mathfrak{M}_1, w_1)$ from $(\mathfrak{M}_2, w_2)$ if $\mathfrak{M}_1, w_1 \models \varphi$ and $\mathfrak{M}_2, w_2 \not\models \varphi$
- two formulas φ and ψ are said to be (semantically) equivalent, notation $\varphi \equiv \psi$, if any state in any model which satisfies one of them also satisfies the other

global truth

if φ is true at every state of the model $\mathfrak{M}$, we write $\mathfrak{M} \models \varphi$, and say φ holds throughout $\mathfrak{M}$, or φ is globally true in $\mathfrak{M}$:

$\mathfrak{M} \models \varphi$ if and only if $\mathfrak{M}, w \models \varphi$, for all points $w \in W$

validity

- let $\mathfrak{F} = (W, R)$ be a frame. a formula φ over Ω is defined to be valid in a frame $\mathfrak{F} = (W, R)$, notation $\mathfrak{F} \models \varphi$, iff φ holds throughout all models based on $\mathfrak{F}$:

$$\mathfrak{F} \models \varphi \quad \text{if and only if} \quad \mathfrak{F}, V \models \varphi, \quad \text{for all valuations } V : \Omega \to \mathbf{2}^W$$

- let K be a class of frames. we write $\mathsf{K} \models \varphi$ if $\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathsf{K}$
- if φ is valid in all frames, φ is said to be universally valid:

$$\models \varphi \qquad \text{if and only if} \qquad \mathfrak{F} \models \varphi, \quad \text{for all frames } \mathfrak{F}$$

three notions of semantics

- local truth:

  $\mathfrak{M}, w \models \varphi$ — φ is true in point w of model $\mathfrak{M}$

- global truth:

  $\mathfrak{M} \models \varphi$ — φ is true throughout model $\mathfrak{M}$:
  $\mathfrak{M}, w \models \varphi$ for all points $w \in W$

- validity:

  $\mathfrak{F} \models \varphi$ — φ is valid in frame $\mathfrak{F}$:
  $(\mathfrak{F}, V) \models \varphi$ for all valuations V

  $\mathsf{K} \models \varphi$ — φ is valid in frame class K:
  $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F} \in \mathsf{K}$

  $\models \varphi$ — φ is universally valid, φ is a tautology:
  $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F}$

semantics: some pitfalls

we have:

$$\mathfrak{M}, w \not\models \varphi \implies \mathfrak{M}, w \models \neg\varphi$$

$$\mathfrak{M}, w \models \varphi \vee \psi \implies \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi$$

but NOT:

$$\mathfrak{M} \not\models \varphi \implies \mathfrak{M} \models \neg\varphi$$

$$\mathfrak{M} \models \varphi \vee \psi \implies \mathfrak{M} \models \varphi \text{ or } \mathfrak{M} \models \psi$$

and also NOT:

$$\mathfrak{F} \not\models \varphi \implies \mathfrak{F} \models \neg\varphi$$

$$\mathfrak{F} \models \varphi \vee \psi \implies \mathfrak{F} \models \varphi \text{ or } \mathfrak{F} \models \psi$$

examples of (in)valid axioms and rules

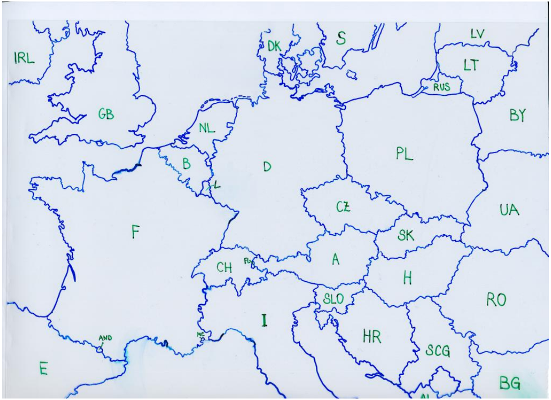
- all propositional tautologies are universally valid
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is universally valid
- modus ponens: if $\models \varphi \to \psi$ and $\models \varphi$, then $\models \psi$
- necessitation: if $\models \varphi$, then $\models \Box\varphi$
- $\not\models \varphi \to \Box\varphi$
- $\models \Box(\varphi \land \psi) \leftrightarrow \Box\varphi \land \Box\psi$
- $\models \Diamond(\varphi \lor \psi) \leftrightarrow \Diamond\varphi \lor \Diamond\psi$
- $\models \Box\varphi \lor \Box\psi \to \Box(\varphi \lor \psi)$
- $\not\models \Box(\varphi \lor \psi) \to \Box\varphi \lor \Box\psi$
- $\models \Diamond(\varphi \land \psi) \to \Diamond\varphi \land \Diamond\psi$
- $\not\models \Diamond\varphi \land \Diamond\psi \to \Diamond(\varphi \land \psi)$

example 1

$$\mathfrak{M} = (W, R, V) \qquad V(p) = \{2, 3\}$$
$$W = \{1, 2, 3, 4, 5\} \qquad V(q) = W$$
$$R = \{(n, m) \mid m = n + 1\} \qquad V(r) = \varnothing$$

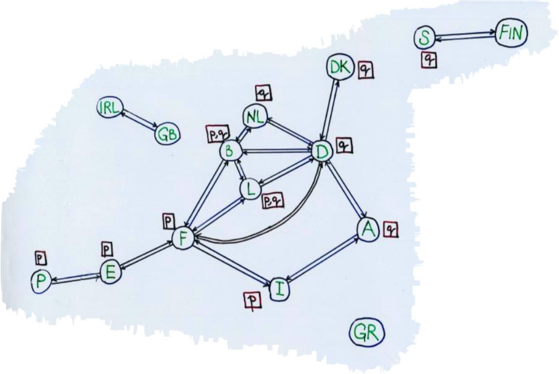
$$\mathfrak{M}, 1 \models \Diamond\Box p$$
$$\mathfrak{M}, 1 \not\models \Diamond\Box p \rightarrow p$$
$$\mathfrak{M}, 2 \models \Diamond p \wedge \neg r$$
$$\mathfrak{M}, 1 \models q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$$
$$\mathfrak{M}, 1 \not\models q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q))))$$
$$\mathfrak{M}, 1 \models q \wedge \Box(q \wedge \Box(q \wedge \Box(q \wedge \Box(q \wedge \Box q))))$$
$$\mathfrak{M} \models \Box q$$

example 2



let $\mathfrak{E} = (E, R)$ with E the members of the EU in 1995, and $R \subseteq E \times E$ defined by Rst if and only if s and t share borders.

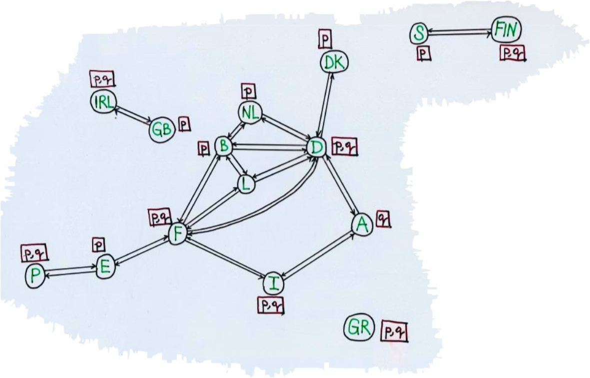
model 1 on EU-frame



let $\mathfrak{M}_1 = (\mathfrak{E}, V_1)$ with $V_1(p)$ the set of EU-members where a Romanic language is an official language, and $V_1(q)$ the set of EU-members where a Germanic language is an official language.

$$\mathfrak{M}_1 \models p \wedge q \rightarrow \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \ \ ?$$

model 2 on EU-frame



now let $\mathfrak{M}_2 = (\mathfrak{E}, V_2)$ with $V_2(p)$ = the EU-members located by the sea and $V_2(q)$ the republics among the EU-members.

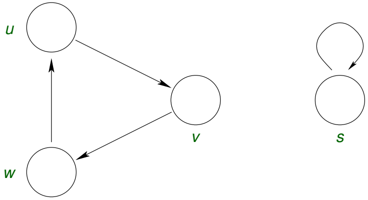
$$\mathfrak{M}_2 \vDash p \wedge q \rightarrow \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \ \ ?$$

example of (in)valid formula in the EU-frame

is the formula $p \to \Box\Diamond p$ valid on the EU-frame ?

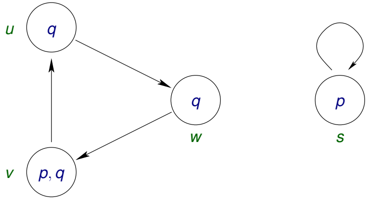
$$\mathfrak{E} \models p \to \Box\Diamond p \quad ?$$

example 3



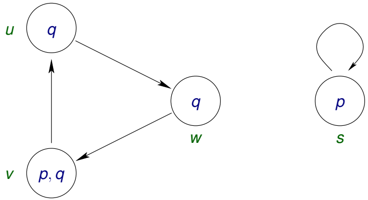
$$\mathfrak{F} = (W, R)$$
$$W = \{u, v, w, s\}$$
$$R = \{(u, v), (v, w), (w, u), (s, s)\}$$

example 3



$$\mathfrak{M} = (\mathfrak{F}, V)$$
$$V(p) = \{v, s\}$$
$$V(q) = \{u, v, w\}$$

example 3

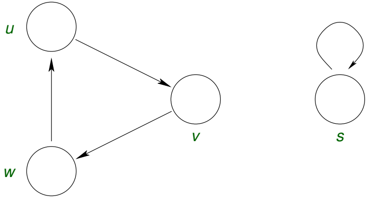


(a) for which worlds $x \in W$ do we have:

(i) $\mathfrak{M}, x \models p \rightarrow \Box p$

(ii) $\mathfrak{M}, x \models \Box p \rightarrow \Diamond q$

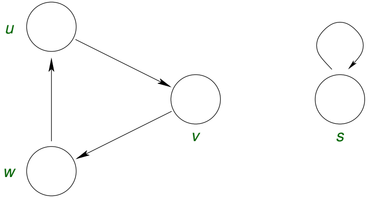
example 3



(b) if possible, give a valuation V_1 on the frame $\mathfrak{F}$ such that for the model $\mathfrak{M}_1 = (\mathfrak{F}, V_1)$ we have:

$$\mathfrak{M}_1 \not\models \Diamond p \to p$$

example 3



(c) if possible, give a valuation V_2 on the frame $\mathfrak{F}$ such that for the model $\mathfrak{M}_2 = (\mathfrak{F}, V_2)$ we have:

$$\mathfrak{M}_2 \not\models \Diamond p \to \Box p$$

example 4

$$\mathfrak{M} = (W, R, V)$$
$$W = \{1, 2, 3, 4, 6, 8, 12, 24\}$$
$$Rxy \text{ iff } x \text{ divides } y \text{ and } x \neq y$$
$$V(p) = \{4, 8, 12, 24\}$$
$$V(q) = \{6\}$$

$\mathfrak{M}, 4 \models \Box p$

$\mathfrak{M}, 6 \models \Box p$

$\mathfrak{M}, 2 \not\models \Box p$

$\mathfrak{M}, 2 \models \Diamond(q \wedge \Box p) \wedge \Diamond(\neg q \wedge \Box p)$

$\mathfrak{M}, 3 \models \Diamond\Box\bot$

$\mathfrak{M} \models \Diamond\Diamond p \rightarrow \Diamond p$

- is $\Diamond\Box\bot$ globally true in $\mathfrak{M}$?
- can you find a valuation V' such that

$$(W, R, V') \not\models \Diamond\Diamond p \rightarrow \Diamond p \ \ ?$$

example 5

$$\begin{aligned} \mathfrak{F} &= (W, R) \\ W &= \{a, b, c, d\} \\ R &= \{(a, b), (b, c), (c, a), (d, a), (d, c)\} \\ V(p) &= \{a, c\} \end{aligned}$$

$(\mathfrak{F}, V), a \models p$ but $(\mathfrak{F}, V) \not\models p$

$(\mathfrak{F}, V) \models p \to \Box^3 p$ but $\mathfrak{F} \not\models p \to \Box^3 p$

$\mathfrak{F} \models \Box^4 p \leftrightarrow \Box p$

modal definability of some elementary frame properties

we have seen that:

1. $p \to \Box\Diamond p$ is valid in all symmetric frames
2. $\Diamond p \to \Box p$ is valid in all deterministic frames
3. $\Box^4 p \leftrightarrow \Box p$ is valid in frames (W, R) with $R^4 = R$

further note that:

- substitution preserves validity: if φ is valid in a frame $\mathfrak{F}$ then so are all its substitution instances φ^σ (next week)
- so we can fill in any modal formula φ for the variable p above, and still have that $\varphi \to \Box\Diamond\varphi$ is valid in all symmetric frames, etc.
- what about the reverse directions of 1,2,3?
  - is symmetry of R implied by $(W, R) \models \varphi \to \Box\Diamond\varphi$, etc.?
- they are true as well! (next week)