Natural Deduction Rules

James Pustejovsky

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Brandeis University

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5.1 Natural deduction rules for $\wedge, \to, \vee$

There are two rules for each connective. The rules reflect the meanings of the connectives.
The easiest is $\wedge$ ('and').

Rules for $\wedge$

- ($\wedge$-introduction, or $\wedge I$) To introduce a formula of the form $A \wedge B$, you have to have already introduced A and B.

1	A	we proved this...
	$\vdots$	(other junk)
2	B	and this...
3	$A \wedge B$	$\wedge I(1,2)$

The line numbers are essential for clarity.

Rules for $\wedge$ ctd.

- ($\wedge$-elimination, or $\wedge E$) If you have managed to write down $A \wedge B$, you can go on to write down A and/or B.

1	$A \wedge B$	we proved this somehow
2	A	$\wedge E(1)$
3	B	$\wedge E(1)$

Rules for $\vee$

- ($\vee$-introduction, or $\vee I$)
  To prove $A \vee B$, prove A, or (if you prefer) prove B.

1	A	proved this somehow
2	$A \vee B$	$\vee I(1)$

  B can be any formula at all!

1	B	proved this somehow
2	$A \vee B$	$\vee I(1)$

  A can be any formula at all.

Rules for $\vee$, ctd.

- ($\vee$-elimination, or $\vee E$) To prove something from $A \vee B$, you have to prove it by assuming A, AND prove it by assuming B. (This is arguing by cases.)

1	$A \vee B$				we got this somehow
2	A	ass	5	B	ass
3	$\vdots$	the 1st proof	6	$\vdots$	the 2nd proof
4	C	we got it	7	C	we got it again
8	C				$\vee E(1,2,4,5,7)$

The assumptions A, B are not usable later, so are put in (side-by-side) boxes. *Nothing inside the boxes can be used later.*

Rules for $\rightarrow$

- ($\rightarrow$-introduction, $\rightarrow I$: 'arrow-introduction') To introduce a formula of the form $A \rightarrow B$, you *assume* A and then prove B.

  During the proof, you can use A as well as anything already established. But *you can't use A or anything from the proof of B from A later on* (because it was based on an extra assumption). So we isolate the proof of B from A, in a *box:*

1	A	ass
	$\langle$the proof$\rangle$	hard struggle
2	B	we made it!
3	$A \rightarrow B$	$\rightarrow I(1,2)$

  Nothing inside the box can be used later.

In natural deduction, boxes are used when we make additional assumptions. The first line inside a box should always be labelled 'ass' (assumption) — with one exception, coming later (p. 212).

Rules for $\rightarrow$, ctd.

- ($\rightarrow$-elimination, or $\rightarrow E$) If you have managed to write down A and $A \rightarrow B$, in either order, you can go on to write down B. (This is modus ponens.)

1	$A \rightarrow B$	we got this somehow…
	$\vdots$	other junk
2	A	and this too…
3	B	$\rightarrow E(1,2)$

5.3 Rules for $\neg$

This is the trickiest case. Also, $\neg$ has three rules! The first two treat $\neg A$ like $A \to \bot$.

- ($\neg$-introduction, $\neg I$) To prove $\neg A$, you assume A and prove $\bot$. As usual, you can't then use A later on, so enclose the proof of $\bot$ from assumption A in a box:

1	A	ass
2	$\vdots$	more hard work, oh no
3	$\bot$	we got it!
4	$\neg A$	$\neg I(1, 3)$

Rules for $\neg$, ctd.

- ($\neg$-elimination, $\neg E$)
  From A and $\neg A$, deduce $\perp$:

1	$\neg A$	proved this somehow...
2	$\vdots$	junk
3	A	...and this
4	$\perp$	$\neg E(1,3)$

- ($\neg\neg$-elimination, $\neg\neg$):
  From $\neg\neg A$, deduce A. (See example 5.8.)